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# Planar rooted trees and non-associative exponential series

L. Gerritzen

*Ruhr-Universität Bochum, Germany*

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**Abstract**

The grafting of finite planar reduced rooted trees is a system of $m$-ary operations, $m \geqslant 2$, on the set **PRT** of those trees.

We consider power series with rational coefficients in a single variable $x$ where the set of monomials is $\{1\} \dot{\cup}$ **PRT**, 1 is the empty tree and $x$ denotes the tree with a single node. Grafting induces a string $(\cdot_m)_{m \geqslant 2}$ of $m$-ary multiplications on this power series algebra $\mathbb{Q}\{\{x\}\}_\infty$.

We show that for any natural number $k \geqslant 2$ there is a unique power series $\exp_k(x) \in \mathbb{Q}\{\{x\}\}_\infty$ such that

$$\big(\exp_k(x)\big)^k = \exp_k(kx)$$

and $\exp_k(0) = \exp_k'(0) = 1$, where $\exp_k'(x)$ is the formal derivative of $\exp_k(x)$ with respect to $x$. We suggest to call it the $k$-ary non-associative exponential series.

It follows that

$$\exp_k'(x) = \exp_k(x).$$

Also it is shown that there is a unique generic exponential $\exp(q, x)$ where the coefficients are in the field $\mathbb{Q}(q)$ of rational functions in a single variable $q$ over $\mathbb{Q}$ such that

$$\exp(k, x) = \exp_k(x) \quad \text{if } k \in \mathbb{N}_{\geqslant 2}.$$

The limit $\lim_{q \to \infty} \exp(q, x)$ exists and is equal to the classical-looking series

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} \in \mathbb{Q}\{\{x\}\}_\infty.$$

*E-mail address:* lothar.gerritzen@ruhr-uni-bochum.de.

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For any tree $t \in \mathbf{PRT}$ the coefficient

$$\mathrm{coeff}_t\big(\exp(q,x)\big)$$

of $\exp(q,x)$ relative to $t$ is of the form $\frac{[t]}{[n-1]!}$ where $n = \deg(t)$, $[n-1]!$ is the $q$-factorial of $n-1$ and $[t]$ is a polynomial in $q$ for which a closed formula is derived.

As the topic of this paper is in the meeting point of combinatorics of trees and non-associative algebra, one can expect further interesting cross-relations.



## Introduction

For any field $K$ there is a power series algebra $K\{\{x\}\}_\infty$ with unit relative to a string $(\cdot_m)_{m\geqslant 2}$ of $m$-ary multiplications with coefficients in $K$ for which the system of monomials $\neq 1$ in $K\{\{x\}\}_\infty$ can be identified with the set **PRT** of isomorphism classes of finite planar reduced rooted trees and the $m$-ary operation on these trees is given by the $m$-ary grafting of planar rooted trees.

For any natural number $k \geqslant 2$ we show that there is a unique power series

$$\exp_k(x)$$

in $\mathbb{Q}\{\{x\}\}_\infty$ such that

$$\big(\exp_k(x)\big)^k = \exp_k(kx)$$

and $\mathrm{ord}(\exp_k(x) - (1+x)) \geqslant 2$. Moreover the derivative $\frac{\mathrm{d}}{\mathrm{d}x}(\exp_k(x))$ is equal to $\exp_k(x)$.

The first terms of $\exp_k(x)$ are given by the expression

$$\begin{aligned}\exp_k(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{k+1}\frac{1}{3!}\left((k-2)x^3 + \frac{3}{2}x^2\cdot x + \frac{3}{2}x\cdot x^2\right)\\ + \frac{1}{(k+1)(k^2+k+1)}\frac{1}{4!}\big[(k+1)(k-2)(k-3)x^4\\ + 2(k+1)(k-2)\big(x^2\cdot x\cdot x + x\cdot x^2\cdot x + x\cdot x\cdot x^2\big)\\ + 2(k-2)\big(x^3\cdot x + x\cdot x^3\big) + 3(k+1)x^2\cdot x^2\\ + 3\big(\big(x^2\cdot x\big)x + \big(x\cdot x^2\big)\cdot x + x\big(x^2\cdot x\big) + x\big(x\cdot x^2\big)\big)\big] + \text{higher terms.}\end{aligned}$$

The case $k = 2$ has been considered in [1]. The coefficient of $\exp_2(x)$ relative to $t \in \mathbf{PRT}$ is non-zero only if $t$ is a binary tree. Therefore $m$-ary multiplications do not occur in $\exp_2(x)$ for $m > 2$. Questions of Doron Zeilberger have stimulated the search for series $\exp_k(x)$ for $k > 2$.

In [3] a combinatorial formula for the coefficients $a_t = a_t(2)$ of the 2-ary exponential $\exp_2(x)$ with respect to a tree $t$ was obtained as follows: If $\hat{a}_t := n!\omega(n) \cdot a_t$ with

$$\omega(n) = \frac{2^{n-1}\prod_{i=2}^{n-1}(2^i - 1)}{n!}$$

then $\hat{a}_t$ is a natural number and

$$\hat{a}(t) = \prod_{a \in I(t)} \binom{n(a) - 2}{n_1(a) - 1}_M$$

where $I(t)$ is the set of inner nodes of $t$ and $n(a)$ is the degree of $t_{\leqslant a}$ where $t_{\leqslant a}$ is the tree below $a$ which is defined to consist of all nodes $b$ for which the simple path from $b$ to the root of $t$ is passing through $a$. Also $n_1(a)$ is the degree of the left factor $s_1$ of $t_{\leqslant a}$ (such that $t_{\leqslant a} = s_1 \cdot s_2$).

Here $\binom{n}{m}_M$ denotes the Mersenne binomial which is defined to be the value of the rational function

$$\frac{[n]!}{[m]![n-m]!}$$

at the place $q = 2$ and $[n]!$ is the $q$-factorial of $n$, see [5] or Section 4.

From the fact that the classical series associated to $\exp_2(x)$ is the Taylor expansion $e^x$ of the exponential function at the point 0, it follows that

$$\sum_{\deg(t)=n} \hat{a}(t) = \omega(n).$$

D. Zeilberger asked if there is an extension of this formula to numbers different from 2. It will be shown in a subsequent paper that this is indeed the case. It can be seen as an application to the combinatorics of trees.

The inverse of $\exp_k(x)$ with respect to the composition of series is $\log_k(1+x)$ for which the functional equation is given by

$$\log_k\bigl((1+x)^k\bigr) = k \cdot \log_k(1+x).$$

These series will be studied in a subsequent article.

Let $\mathbb{Q}(q)$ be the field of rational functions in a variable $q$ over $\mathbb{Q}$. There is a power series $\exp(q, x) \in \mathbb{Q}(q)\{\{x\}\}_\infty$ such that

$$\exp(k, x) = \exp_k(x)$$

for $k \in \mathbb{N}_{\geqslant 2}$. It is called the generic exponential series.

It is similarly possible to define a generic logarithm $\log(q, 1+x)$ for which

$$\exp\bigl(q, \log(q, 1+x)\bigr) = 1+x,$$
$$\log\bigl(q, \exp(q, x)\bigr) = x.$$

Recently it was suggested by J.-L. Loday to extend the results about the canonical projection onto the Lie polynomials, see [10, Chapter (3.2), pp. 57–61], to the non-associative setting [7]. One can expect to interpret $\log(q, \mathrm{Id})$ relative to the convolution product as a generic first Eulerian idempotent $e^{(1)}$ and to be able to construct higher idempotents $e^{(t)}$ for any tree $t$ in **PRT**. There should be relations between these idempotents and the $q$-idempotents found in [2].

In this paper we start the systematic study of properties of $\exp(q, x)$ and derive formulas for the coefficients $a(q, t)$ relative to $t \in \mathbf{PRT}$ of $\exp(q, x)$ by methods of tree combinatorics. If $t = t_1 \cdot \cdots \cdot t_m \in \mathbf{PRT}$ and $n = \deg(t)$, then we get the recursive formula

$$\bigl(q^n - q\bigr) \cdot a(q, t_1 \cdot \cdots \cdot t_m) = \binom{q}{m} \cdot a(q, t_1) \cdot \cdots \cdot a(q, t_m).$$

Also

$$a(q, x) = \frac{[t]}{[n-1]!}$$

where $[t]$ is a polynomial in $\mathbb{Q}[q]$, see Section 4, $[n-1]! = \prod_{i=1}^{n-1} [i]$ is the $q$-factorial of $(n-1)$ with $[i] = \sum_{j=0}^{i-1} q^j$.

Let $\mathbb{Q}(q)[[x]]$ denote the commutative, associative algebra of power series over $\mathbb{Q}(q)$ in one variable $x$. There is a canonical homomorphism

$$\mathrm{can}: \mathbb{Q}(q)\{\{x\}\}_\infty \to \mathbb{Q}(q)[[x]]$$

which maps a monomial $t \in \mathbf{PRT}$ of degree $n$ onto $x^n$ and we obtain that $\mathrm{can}(\exp(q, x))$ is equal to the Euler exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

As a consequence we get summation formulas

$$\sum_{t \in \mathbf{PRT}(n)} [t] = \frac{[n-1]!}{n!}$$

for which a combinatorial explanation would be of interest.

A functional equation for $\exp(q, x)$ is derived in Section 5 and it is shown that $\lim_{q\to\infty} \exp(q, x)$ exists and is equal to $\sum_{m=0}^{\infty} \frac{x^m}{m!}$.

In Section 1 we fix notations about the system **PRT** of finite, planar reduced rooted trees and the grafting operations.

The algebra $K\{\{x\}\}_\infty$ of power series with monomials in $\mathbf{PRT}' = \{1_{\mathbf{PRT}}\} \cup \mathbf{PRT}$ over a field $K$ is introduced in Section 2 where $1_{\mathbf{PRT}}$ denotes the empty tree. The formal derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ on $K\{\{x\}\}_\infty$ is introduced in Proposition 2.1.

In Section 3 we investigate the fundamental properties of $\exp_k(x)$. For any finite, reduced rooted tree $R$ we define the $q$-polynomial $[R] \in \mathbb{Q}[q]$ in Section 4 and derive a closed formula in Proposition 4.3. In Section 5, $q$-polynomials are used to define the generic exponential series. We give a functional equation for the generic exponential and consider the special series $\exp(\infty, x)$, $\exp(0, x)$ and $\exp(1, x)$.

In Section 6 we are drawing conclusions from the fact that the classical series of $\exp(q, x)$ is Euler's series. Some open problems are mentioned in Section 7. There is the general question how to extend classical functions and formulas to the non-associative setting.

## 1. Planar reduced rooted trees

For any graph $G$ and any node $\eta$ of $G$, we denote by $\mathrm{val}_G(\eta)$ the number of edges of $G$ which are incident with $\eta$ and call it the valence of $\eta$ in $G$.

A rooted tree $R$ is a pair $(T, \eta)$ where $T$ is a tree and $\eta$ a node of $T$ which is called the root of $R$ and which will be denoted by $\rho_R$. For any rooted tree $R$, let $R^0$ be the set of nodes of $R$ and $L(R) = \{\eta \in R^0\colon\ \mathrm{val}_R(\eta) \leqslant 1\}$. $L(R)$ is called the set of leaves of $R$.

If $\sharp R^0 > 1$, then $\mathrm{val}_R(\eta) \geqslant 1$ for all $\eta \in R^0$ and $L(R) = \{\eta \in R^0\colon\ \mathrm{val}_R(\eta) = 1\}$.

**Definition 1.1.** A rooted tree $R$ is called reduced, if

(i) $\mathrm{val}_R(\rho_R) \neq 1$,
(ii) $\mathrm{val}_R(\eta) \neq 2$ for any node $\eta$ of $R$ which is different from the root $\rho_R$ of $R$.

The rooted tree which contains a single node is reduced and will be denoted by $x$. Then $\sharp L(x) = 1$.

If $R$ is a rooted tree which contains more than one node, then $R$ is reduced, if $\mathrm{val}_R(\rho_R) \geqslant 2$ and $\mathrm{val}_R(\eta) \geqslant 3$ for all $\eta \in R^0 - L(R)$, $\eta \neq \rho_R$.

For any rooted tree $R$ and any node $\eta$ of $R$, we denote by $R(\eta)$ the subtree of $R$ whose set of nodes consists of all nodes $\eta'$ of $R$ for which the unique simple path from $\rho_R$ to $\eta'$ is passing through $\eta$. Then $\eta$ is a node of $R(\eta)$ and it will be considered as root of $R(\eta)$ turning $R(\eta)$ into a rooted tree. Then $L(R(\eta)) \subseteq L(R)$.

If $R$ is reduced, then also $R(\eta)$ is reduced for all nodes $\eta$ of $R$.

We denote by $\mathbf{RT}$ the set of isomorphism classes of finite, reduced rooted trees. For any tree $T \in \mathbf{RT}$ we call $\deg(T) := \sharp L(T)$ the degree of $T$.

The set of trees $T$ in $\mathbf{RT}$ whose degree is $n$ is denoted by $\mathbf{RT}(n)$.

For any $T \in \mathbf{RT}$ we call $\mathrm{ar}(T) := \mathrm{val}(\rho_T)$ the arity of $T$. For any node $\eta$ in $T$, we call $\mathrm{ar}_T(\eta) := \mathrm{ar}(T(\eta))$ the arity of $\eta$ in $T$. Thus $\mathrm{ar}_T(\eta) = \mathrm{val}_T(\eta) - 1$ if $\eta \neq \rho_T$.

Let $T \in \mathbf{RT}(n)$, $n^0(T)$ be the number of nodes in $T$ and $\bar{n}(T)$ the number of edges in $T$. Then it is easy to see that

$$n^0(T) = \bar{n}(T) + 1.$$

Also $\mathrm{ar}_T(\eta) = 0$ if and only if $\eta$ is a leaf of $T$ and

$$\sum_{\eta \in T^0} \mathrm{ar}_T(\eta) = n^0(T) - 1$$

because the right-hand side is counting the set of edges of $T$. It follows that

$$\sum_{\eta \in T^{(\mathrm{in})}} \bigl(\mathrm{ar}_T(\eta) - 1\bigr) = \deg(T) - 1$$

where $T^{(\mathrm{in})}$ is the set of inner nodes of $T$, $T^{(\mathrm{in})} = T^0 - L(T)$.

The set of trees $T$ in $\mathbf{RT}$ with arity $m$ is denoted by $\mathbf{RT}_m$. Then $\mathbf{RT}_0 = \{x\}$, where $x$ denotes the tree with a single node and $\mathbf{RT}_1 = \emptyset$.

Let $R$ be a reduced rooted tree and $\leqslant$ be a total order on the set $L(R)$ of leaves of $R$.

**Definition 1.2.** The pair $(R, \leqslant)$ is called planar reduced rooted tree, if for all nodes $\eta$ of $R$ the set $L(R(\eta))$ is an interval in $L(R)$ which means there are $\lambda_1, \lambda_2 \in L(R)$ such that $L(R(\eta)) = \{\lambda \in L(R)\colon \lambda_1 \leqslant \lambda \leqslant \lambda_2\}$.

There is also a geometric definition of planarity of trees which is widely used, for example in Operad Theory, see [9].

One defines planar embeddings of trees as geometric realizations into the plane $\mathbb{R}^2$. Two realizations are called equivalent if they are related by a continuous isotopy of the plane. A geometric planar tree is a tree together with an equivalence class of geometric realizations, see for instance [9, Part II, (1.5), p. 50].

Any geometric realization of a reduced rooted tree $T$ with more then one node is isotopic to a special one in the following sense: the leaves of $T$ are on the real line $\mathbb{R}$, all the inner nodes of $T$ are in the upper half plane and the simple path from the smallest leaf of $T$ under the realization on $\mathbb{R}$ has no common edge with the simple path to the largest leaf of $T$ on $\mathbb{R}$.

It is easy to check that two special realizations of a reduced rooted tree $T$ are isotopic if and only if the order on the set $L(T)$ induced by the order relation on $\mathbb{R}$ with respect to the geometric realizations are equal.

This shows that the geometric definition of planar reduced rooted trees agrees with the combinatorial definition in 1.2.

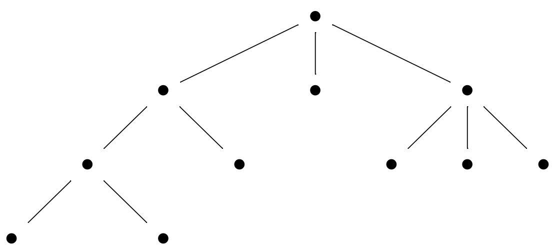
We denote by $\mathbf{PRT}$ the set of isomorphism classes of finite, planar reduced rooted trees. Let $m \geqslant 2$ and $R_1, \ldots, R_m \in \mathbf{PRT}$. There is a unique $R \in \mathbf{PRT}$ with the following properties:

(i) The arity of $R$ is $m$.

(ii) There are nodes $\eta_1, \ldots, \eta_m, \eta_i \neq \eta_j$ for $i \neq j$, such that $\eta_i$ is adjacent to the root $\rho_R$ of $R$, $R(\eta_i)$ is isomorphic with $R_i$ for all $i$.
(iii) $L(R)$ is the disjoint union of $L(R_1), \ldots, L(R_m)$.
(iv) For any $\eta, \eta' \in L(R), \eta \in L(R_i), \eta' \in L(R_j)$ one has $\eta \leqslant_R \eta'$ if and only if $i < j$ or $i = j$ and $\eta \leqslant_{R_i} \eta'$.

The tree $R$ in **PRT** is denoted by $\cdot_m(R_1, \ldots, R_m)$ or $(R_1 \cdot R_2 \cdot \cdots \cdot R_m)$ and is called the planar grafting of $R_1, \ldots, R_m$. The map $\cdot_m$ is an $m$-ary composition law on **PRT**. The arity of $R = R_1 \cdot \cdots \cdot R_m$ is $m$ and the degree of $R$ is $\sum_{i=1}^m \deg(R_i)$. We denote by $R^m$ the $m$-fold grafting $\cdot_m(R, \ldots, R)$ of $R$.

The following figure shows a planar reduced rooted tree $T$ of degree 7 and arity 3 for which the node on top is the root of $T$



It is isomorphic to $((x^2 \cdot x) \cdot x \cdot x^3)$.

Let $\mathbf{PRT}_m(n) = \mathbf{PRT}_m \cap \mathbf{PRT}(n)$ be the set of trees in **PRT** of arity $m$ and degree $n$.

$$\mathbf{PRT}(1) = \{x\},$$
$$\mathbf{PRT}(2) = \{x^2\},$$
$$\mathbf{PRT}(3) = \{x^3, x^2 \cdot x, x \cdot x^2\},$$
$$\mathbf{PRT}(4) = \mathbf{PRT}_2(4) \cup \mathbf{PRT}_3(4) \cup \mathbf{PRT}_4(4), \quad \text{and}$$
$$\mathbf{PRT}_4(4) = \{x^4\},$$
$$\mathbf{PRT}_3(4) = \{x^2 \cdot x \cdot x, x \cdot x^2 \cdot x, x \cdot x \cdot x^2\},$$
$$\mathbf{PRT}_2(4) = \{x^3 \cdot x, (x^2 \cdot x) \cdot x, (x \cdot x^2) \cdot x, x \cdot x^3, x \cdot (x^2 \cdot x), x \cdot (x \cdot x^2), x^2 \cdot x^2\}.$$

Thus $\sharp\mathbf{PRT}(4) = 11$. One can show that $\sharp\mathbf{PRT}(5) = 45$, $\sharp\mathbf{PRT}(6) = 197$, $\sharp\mathbf{PRT}(7) = 903$.

The generating series of **PRT** relative to the grading given by the degree-function is

$$\frac{1}{4}\left(1 + x - \sqrt{1 - 6x + x^2}\right) \in \mathbb{N}[[x]].$$

This formula is proved below, see Proposition 2.3.

For any numeric partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ we denote by $\binom{|\lambda|}{\lambda}$ the multinomial

$$\binom{n}{\lambda_1, \ldots, \lambda_m} = \frac{n!}{\lambda_1! \cdots \lambda_m!}, \quad \text{if } n = |\lambda| = \lambda_1 + \cdots + \lambda_m.$$

Let $T = (R, \leqslant)$ be a planar tree in **PRT**. Then $R$ is called the tree in **RT** underlying $T$; it is denoted by $|T|$. Let now $R$ be a tree in **RT**. For any inner node $\eta$ of $R$ of arity $m$, let $\{\eta_1, \ldots, \eta_m\}$ be the nodes in $R(\eta)$ adjacent with $\eta$. Then $\eta_i$ is called equivalent to $\eta_j$ if $R(\eta_i)$ is isomorphic with $R(\eta_j)$. The numeric partition of this equivalence relation is denoted by $\sigma(\eta)$; it is a partition of $m = \mathrm{ar}(\eta)$.

**Proposition 1.3.** *Let $R \in \mathbf{RT}$. Then $\pi(R) := \sharp\{T \in \mathbf{PRT}\colon |T| = R\} = \prod_{\eta \in R^{(\mathrm{in})}} \binom{|\sigma(\eta)|}{\sigma(\eta)}$, where $R^{(\mathrm{in})}$ is the set of inner nodes of $R$.*

**Proof.** Let $T_1, \ldots, T_m \in \mathbf{PRT}$ and $\tau$ be a permutation of $\{1, 2, \ldots, m\}$. Then $T_1 \cdot T_2 \cdot \cdots \cdot T_m = T_{\tau(1)} \cdot \cdots \cdot T_{\tau(m)}$, with respect to planar grafting if and only if $T_{\tau(i)} = T_i$ for all $1 \leqslant i \leqslant m$. It follows that $\sharp\{T_{\tau(1)} \cdot \cdots \cdot T_{\tau(m)}\colon \tau \text{ permutation of } \{1, \ldots, m\}\} = \binom{|\sigma(\rho_R)|}{\sigma(\rho_R)}$ if $R$ is the rooted tree underlying $T_1 \cdot T_2 \cdot \cdots \cdot T_m$. The result follows with induction on the degree of $R$. $\square$

### 1.1. *Adjunction of a unit* $1_{\mathbf{PRT}}$

Let $1_{\mathbf{PRT}}$ be an element not contained in **PRT** and let $\mathbf{PRT}' := \{1_{\mathbf{PRT}}\} \cup \mathbf{PRT}$. One can think of $1_{\mathbf{PRT}}$ as the empty tree and define $\deg(1_{\mathbf{PRT}}) = 0$.

We will extend the definition of the $m$-ary operation $\cdot_m$ on **PRT** to $\mathbf{PRT}'$. For $R_1, R_2 \in \mathbf{PRT}'$ we put

$$R_1 \cdot R_2 = \begin{cases} R_1 & \text{if } R_2 = 1_{\mathbf{PRT}}, \\ R_2 & \text{if } R_1 = 1_{\mathbf{PRT}}. \end{cases}$$

Let now $m > 2$ and $R_1, \ldots, R_m \in \mathbf{PRT}'$. We define $R = R_1 \cdot R_2 \cdot \cdots \cdot R_m$ inductively on $m$ by setting

$$R = R_1 R_2 \cdot \cdots \cdot R_{i-1} R_{i+1} \cdot \cdots \cdot R_m$$

if $R_i = 1_{\mathbf{PRT}}$.

## 2. Planar tree power series

Let $K$ be a field and $\mathbf{A} = K^{\mathbf{PRT}'}$ the $K$-vectorspace of all $K$-valued maps $f : \mathbf{PRT}' \to K$. For any $t \in \mathbf{PRT}'$ we denote by $\mathrm{coeff}_t(f)$ the value of $f$ at $t$ and call it the coefficient of $f$ relative to $t$.

Let now $f_1, \ldots, f_m \in \mathbf{A}$. We define $f = f_1 \cdot \cdots \cdot f_m$ by setting

$$\mathrm{coeff}_t(f) = \sum \mathrm{coeff}_{t_1}(f_1) \cdot \mathrm{coeff}_{t_2}(f_2) \cdot \cdots \cdot \mathrm{coeff}_{t_m}(f_m)$$

where the sum is extended over all $(t_1, \ldots, t_m) \in (\mathbf{PRT}')^m$ with $t = t_1 \cdot t_2 \cdot \cdots \cdot t_m$. Thus we have defined for all $m \geqslant 2$ an $m$-ary composition law on $\mathbf{A}$ which is obviously $K$-multilinear.

We think of $(\mathbf{A}, \cdot_m, m \geqslant 2)$ as a $K$-algebra system with a string of $m$-ary multilinear operations; it will also be denoted by $\mathbf{A}$ or by $K\{\{x\}\}_\infty$ indicating that there is an infinity of independent operations. It will be called the algebra of planar tree power series in $x$ over $K$.

For any $f \in K\{\{x\}\}_\infty$ let $\operatorname{ord}(f) := \min\{\deg(t)\colon\ t \in \mathbf{PRT}', \operatorname{coeff}_t(f) \neq 0\}$. As usual $\operatorname{ord}(0) = \infty$. It defines an algebra topology on $K\{\{x\}\}_\infty$ for which a system of neighbourhoods of 0 is given by the family $(U_n)_{n \geqslant 0}$ with $U_n = \{f \in K\{\{x\}\}_\infty\colon \operatorname{ord}(f) \geqslant n\}$.

A $K$-linear continuous map $\theta : \mathbf{A} \to \mathbf{A}$ is called a derivation on $\mathbf{A}$, if

$$\theta(f_1 \cdot f_2 \cdot \cdots \cdot f_m) = \sum_{i=1}^{m} \bigl(f_1 \cdot \cdots \cdot f_{i-1} \cdot \theta(f_i) \cdot f_{i+1} \cdot \cdots \cdot f_m\bigr)$$

for all $m \geqslant 2$ and all $f_1, \ldots, f_m \in \mathbf{A}$.

**Proposition 2.1.** *There is a unique derivation $\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbf{A}$ such that $\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$*

**Proof.** (1) First we define $\frac{\mathrm{d}}{\mathrm{d}x}(t)$ for all $t \in \mathbf{PRT}'$ inductively on $\deg(t) = n$. We put $\frac{\mathrm{d}}{\mathrm{d}x}(1) = 0$. If $n > 1$ and $t = t_1 \cdot t_2 \cdot \cdots \cdot t_m$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(t) := \sum_{i=1}^{m} t_1 \cdot \cdots \cdot t_{i-1} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(t_i) \cdot t_{i+1} \cdot \cdots \cdot t_m.$$

There is a $K$-linear extension of this map onto the subalgebra $K\{x\}_\infty$ of polynomials in $x$ which has $\mathbf{PRT}'$ as $K$-basis. Then one checks that $\frac{\mathrm{d}}{\mathrm{d}x}$ is a derivation on $K\{x\}_\infty$.

(2) One can extend $\frac{\mathrm{d}}{\mathrm{d}x}$ on $K\{x\}_\infty$ continuously to a derivation on $\mathbf{A}$.

(3) If $\theta$ is another derivation on $\mathbf{A}$ with $\theta(x) = 1$, then also $\hat{\theta} = \theta - \frac{\mathrm{d}}{\mathrm{d}x}$ is a derivation on $P$ and $\hat{\theta}(x) = 0$. One can show that $\operatorname{ord}(\hat{\theta}(f)) \geqslant \operatorname{ord}(f)$ for any $f \in \mathbf{A}$. It follows that $\hat{\theta}$ is continuous and $\hat{\theta}(g) = 0$ for all $g \in K\{x\}_\infty$. Then $\hat{\theta} = 0$ and $\theta = \frac{\mathrm{d}}{\mathrm{d}x}$. This proves the uniqueness of $\frac{\mathrm{d}}{\mathrm{d}x}$. □

**Remark.** The proof above also shows that for any $g \in \mathbf{A}$ there is a unique continuous derivation $\theta$ on $\mathbf{A}$ such that $\theta(x) = g$.

**Proposition 2.2.** *Let $K[[x]]$ be the classical commutative and associative algebra of power series in one variable $x$ over a field $K$. There is a unique canonical algebra homomorphism*

$$\operatorname{can} : K\{\{x\}\}_\infty \to K[[x]]$$

*which maps $x$ in $K\{\{x\}\}_\infty$ onto $x$ in $K[[x]]$. We call* $\operatorname{can}(f)$ *the classical series of $f \in \mathbf{A}$.*

**Proof.** Define $\mathrm{can}(f)$ for $f \in K\{\{x\}\}_\infty$ by $\sum_{n=0}^{\infty}(\sum_{t\in\mathbf{PRT}(n)} \mathrm{coeff}_t(f))x^n$. One can show that this map can is the unique homomorphism of the proposition. □

**Proposition 2.3.** *Let $f(x) = \sum_{t\in\mathbf{PRT}} t(x) \in \mathbb{Q}\{\{x\}\}_\infty$ and $g(x)$ be the classical series of $f$, then $g(x) = 1/4(1 + x - \sqrt{1 - 6x + x^2})$.*

**Proof.** Obviously $f = x + \sum_{r=2}^{\infty} f^r$ and therefore

$$x + \sum_{r=2}^{\infty} g^r(x) = x + \frac{1}{1 - g(x)} \cdot g^2(x).$$

Thus $2g^2(x) - (1 + x)g(x) - x = 0$ and

$$g(x) = 1/4\bigl(1 + x - \sqrt{1 - 6x + x^2}\bigr). \qquad \square$$

**Remark.** Obviously $g(x)$ is the generating series of the graded set **PRT**. One can compute the first terms of $g(x)$ to be

$$x + x^2 + 3x^3 + 11x^4 + 45x^5 + 197x^6 + 903x^7 + 4279x^8 + 20793x^9 + 103049x^{10} + \text{higher terms.}$$

The coefficients of $g(x)$ are sometimes refered to as super-Catalan numbers [8, Section (2.4)], or [6, Section 8].

## 3. Exponential series

**Proposition 3.1.** *Let $k$ be a natural number $\geqslant 2$. There is a unique series $E_k(x) \in \mathbb{Q}\{\{x\}\}_\infty$ such that*

(i) $\mathrm{ord}(E_k(x) - (1 - x)) \geqslant 2$,
(ii) $(E_k(x))^k = E_k(kx)$.

*Moreover* $\frac{\mathrm{d}}{\mathrm{d}x}(E_k(x)) = E_k(x)$.

$E_k(x)$ is called the $k$-ary exponential series and will also be denoted by $\exp_k(x)$.

**Proof.** (1) Let $F \in \mathbf{A} = K\{\{x\}\}_\infty$, $\mathrm{ord}(F) > 0$ and $E = (1 + F)$. For any $k \in \mathbb{N}$, $k \geqslant 2$, we get $E^k = 1 + \sum_{r=1}^{k} \binom{k}{r} F^r$. If $t \in \mathbf{PRT}_m$, $t = t_1 \cdot t_2 \cdot \cdots \cdot t_m$, then

$$\mathrm{coeff}_t\bigl(E^k\bigr) = \binom{k}{m}\mathrm{coeff}_{t_1}(F) \cdot \cdots \cdot \mathrm{coeff}_{t_m}(F) + k\,\mathrm{coeff}_t(F)$$

if $1 \leqslant m \leqslant k$ and $\mathrm{coeff}_t(E^k) = 0$ if $m > k$.

(2) Define a map $a_k : \mathbf{PRT} \to \mathbb{Q}$ by putting $a_k(x) = 1$ and

$$a_k(t_1 t_2 \cdots t_m) = \frac{\binom{k}{m} a_k(t_1) \cdot \cdots \cdot a_k(t_m)}{k^n - k}$$

if $\deg(t_1 t_2 \cdots t_m) = n \geqslant 2$. Let $F := \sum_{t \in \mathbf{PRT}} a_k(t) \cdot t$ and $E = 1 + F$.

Then $\operatorname{ord}(F) \geqslant 1$ and if

$$t = t_1 t_2 \cdot \cdots \cdot t_m \in \mathbf{PRT}_m$$

then

$$\begin{aligned}
\operatorname{coeff}_t\left(E^k\right) &= \operatorname{coeff}_t\left(\binom{k}{m} \cdot F_m\right) + \operatorname{coeff}_t(k \cdot F), \\
\operatorname{coeff}_t\left(E^k\right) &= \binom{k}{m} \operatorname{coeff}_{t_1}(F) \cdot \cdots \cdot \operatorname{coeff}_{t_m}(F) + k \cdot \operatorname{coeff}_t(F) \\
&= \binom{k}{m} a_k(t_1) \cdot \cdots \cdot a_k(t_m) + k \cdot a_k(t) \\
&= \left(k^n - k\right) a_k(t) + k a_k(t) = k^n a_k(t).
\end{aligned}$$

As $\operatorname{coeff}_t(E(kx) = k^n \operatorname{coeff}_t(E(x)) = k^n a_k(t)$ one obtains

$$E^k(x) = E(kx).$$

(3) Assume that $E^k(x) = E(kx)$ and $\operatorname{ord}(E - (1 + x)) \geqslant 2$, then one gets from the computation in (2) that

$$\operatorname{coeff}_t\left(E^k\right) = \binom{k}{m} \operatorname{coeff}_{t_1}(E) \cdots \operatorname{coeff}_{t_m}(E)$$

from which follows that $a_k(t) = \operatorname{coeff}_t(E)$ for all $t \in \mathbf{PRT}'$.

(4) Let $E = E_k$ and $F = E - 1$. Denote by $F_n$ the homogeneous part of $F$ of degree $n$. Thus $F_n = \sum_{\substack{t \in \mathbf{PRT}, \\ \deg(t) = n}} \operatorname{coeff}_t(F) \cdot t(x)$. Then $F_n(kx) = k^n F_n(x)$ and

$$F_n(kx) = \sum_{r=1}^{k} \binom{k}{r} \left(F^r\right)_n$$

where $(F^r)_n$ is the homogeneous part of degree $n$ of $F^r$.

Now

$$\left(F^r\right)_n = \sum_{\nu \in M(r,n)} F_\nu$$

where $M(r.n) = \{\nu = (\nu_1, \dots, \nu_r)\colon \nu_i \in \mathbb{N}_{\geqslant 1}, |\nu| = \nu_1 + \cdots + \nu_r = n\}$ and $F_\nu := F_{\nu_1} \cdot F_{\nu_2} \cdot \cdots \cdot F_{\nu_r}$ for $\nu = (\nu_1, \dots, \nu_r) \in M(r, n)$. We prove that $F_n' := \frac{\mathrm{d}}{\mathrm{d}x}(F_n) = F_{n-1}$ by induction on $n$.

It is obvious if $n = 1$ as $F_1 = x$. Let now $n > 1$. As

$$\bigl(k^n - k\bigr) F_n(x) = \sum_{v=2}^{k} \binom{k}{r} \bigl(F^r\bigr)_n$$

we get

$$\bigl(k^n - k\bigr) F_n'(x) = \sum_{r=2}^{k} \binom{k}{r} \frac{\mathrm{d}}{\mathrm{d}x}\bigl(\bigl(F^r\bigr)_n\bigr).$$

We will prove in part (5) below that

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigl(\bigl(F^r\bigr)_n\bigr) = r\bigl(\bigl(F^r\bigr)_{n-1} + \bigl(F^{r-1}\bigr)_{n-1}\bigr)$$

for any $2 \leqslant r \leqslant n$. Thus

$$\begin{aligned}\bigl(k^n - k\bigr) F_n'(x) = {} & \frac{\mathrm{d}}{\mathrm{d}x}\bigl(\bigl(k \cdot F^k\bigr)_{n-1}\bigr) + \sum_{r=2}^{k-1} \binom{k}{r} \cdot r\bigl(F^r\bigr)_{n-1} \\ & + \binom{k}{r+1} \cdot (r+1) \cdot \bigl(\bigl(F^r\bigr)_{n-1}\bigr) + 2 \cdot \binom{k}{2} F_{n-1}.\end{aligned}$$

As $\binom{k}{r} \cdot r + \binom{k}{r+1}(r+1) = k\binom{k-1}{r-1} + \binom{k-1}{r} = k\binom{k}{r}$ for $2 \leqslant r \leqslant k-1$, we get

$$\begin{aligned}\bigl(k^n - k\bigr) F_n'(x) &= k \sum_{r=2}^{k} \binom{k}{r} \frac{\mathrm{d}}{\mathrm{d}x}\bigl(F^r\bigr)_{n-1} + \bigl(k^2 - k\bigr) F_{n-1} \\ &= k\bigl(k^{n-1} - k\bigr) F_{n-1}(x) + \bigl(k^2 - k\bigr) F_{n-1}(x) \\ &= \bigl(k^n - k\bigr) F_{n-1}(x)\end{aligned}$$

which proves that

$$F_n'(x) = F_{n-1}(x).$$

(5) Now we prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigl(F^r\bigr)_n = r\bigl(\bigl(F^r\bigr)_{n-1} + \bigl(F^{(r-1)}\bigr)_{n-1}\bigr)$$

if $2 \leqslant r \leqslant k$.

By definition

$$\left(F^r\right)_n = \sum_{\nu \in M(r,n)} F_\nu$$

where $M(r,n) = \{\nu = (\nu_1, \dots, \nu_r)\colon\ \nu_i \in \mathbb{N}_{\geqslant 1}, |\nu| = \nu_1 + \cdots + \nu_r = n\}$ and $F_\nu = F_{\nu_1} \cdot \dots \cdot F_{\nu_r}$.

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}(F_\nu) = \sum_{i=1}^{r} F_{r-e_i}$$

where $e_i = (\delta_{i1}, \dots, \delta_{ir}) = (0, \dots, 0, 1, 0, \dots, 0)$ as by induction hypothesis

$$\frac{\mathrm{d}}{\mathrm{d}x} F_j = F_{j-1}$$

for all $j < n$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}(F_\nu) = \sum_{i=1}^{r} F_{\nu_1} \cdot \dots \cdot F_{\nu_{i-1}} F'_{\nu_1} \cdot F_{\nu_{i+1}} \cdot \dots \cdot F_{\nu_r}.$$

Let $N(r,n) := M(r,n) \times \{1, 2, \dots, r\}$, $N(r,n,i) = \{(\nu, i) \in N(r,n)\colon$ the $i$th component $\nu_i$ of $\nu$ is equal to $1\}$ and $N^*(r,n) = N(r,n) - \bigcup_{i=1}^{r} N(r,n,i)$. Now

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(F^r\right)_n\right) = \sum_{\nu \in N^*(r,n)} F_{\nu - e_i} + \sum_{i=i}^{r} \sum_{\nu \in N(r,n,i)} F_{\nu - e_i}.$$

Let $\mu \in M(r, n-1)$. For any $1 \leqslant i \leqslant r$ there is exactly one $\nu = \nu(\mu, i) = \mu + e_i$ such that $F_{\nu - e_i} = F_\mu$. Then $\nu(\mu, i) \notin N(r, n_i, i)$. It follows that

$$\sum_{\nu \in N^*(r,n)} F_{\nu - e_i} = r \cdot \left(F^r\right)_{n-1}.$$

For any $\nu \in N(r,n,i)$ let $\bigwedge_i(\nu) = (\nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_r) \in N(r-1, n-1)$. Then $F(\nu - e_i) = F_{\bigwedge_i(\nu)}$ for all $\nu \in N(r,n,i)$ and $\bigwedge_i$ is a bijective map from $N(r,n,i)$ onto $N(r-1, n-1)$. Thus $\sum_{i=1}^{r} \sum_{\nu \in N(r,n,i)} F_{\nu - e_i} = r \cdot F_{n-1}$ which proves the claim above. $\square$

## 4. The $q$-polynomial of a rooted tree

Let $\mathbb{Q}(q)$ be the field of rational functions in the variable $q$ over $\mathbb{Q}$.

Denote by $[n]$ the quantum analogue of the natural number $n \geqslant 1$ which is defined to be $\frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$, see for instance [5].

Let $[n]!$ be the $q$-factorial, defined to be $\prod_{i=1}^{n}[i]$ Then $[n]!$ is a polynomial in $\mathbb{Z}[q]$ of degree $\sum_{i=0}^{n-1} i = \binom{n}{2}$.

Let $\left[{n \atop r}\right]$ be the $q$-binomial coefficient, defined to be $\frac{[n]!}{[r]![n-r]!}$. It is known that $\left[{n \atop r}\right] \in \mathbb{Z}[q]$ is a primitive polynomial of degree $r(n-r)$, see [5, Corollary (6.1)].

Let $T \in \mathbf{PRT}(n)$. We will define $[T]$ as element in $\mathbb{Q}[q]$ inductively on the degree $\deg(T)$. If $\deg(T) = 1$, then $T = x$, and $[x] := 1$. Let now $n = \deg(T) > 1$ and $T = T_1 \cdot T_2 \cdot \cdots \cdot T_m$, $n_i = \deg(T_i)$. Then $m \geqslant 2$. Define

$$[T] := \frac{[n-2]!}{[n_1-1]! \cdots [n_m-1]!} \cdot \frac{1}{m(m-1)} \binom{q-2}{m-2} \cdot [T_1] \cdot \cdots \cdot [T_m].$$

**Proposition 4.1.** $[T]$ *is a polynomial in* $\mathbb{Q}[q]$; *it is called the* $q$*-polynomial of* $T$.

**Proof.** We proceed by induction on $n = \deg(T)$. If $T = T_1 \cdot T_2 \cdot \cdots \cdot T_m$ as above, then

$$\frac{[n-2]!}{[n_1-1]! \cdots [n_m-1]!} = \prod_{i=n-m+1}^{n-m} [i]! \left[{n-m \atop n_1-1, \ldots, n_m-1}\right]$$

where $\left[{k \atop k_1, \ldots, k_m}\right] := \frac{[k]!}{[k_1]! \cdots [k_r]!} = \left[{k \atop k_1}\right] \cdot \left[{k-k_1 \atop k_2, \ldots, k_m}\right]$.

By induction on $m$ it follows that $\left[{k \atop k_1, \ldots, k_m}\right]$ is a polynomial in $\mathbb{Z}[q]$, because $q$-binomials are in $\mathbb{Z}[q]$.

As $\deg(T_i) < n$, $[T_i]$ is a polynomial in $q$ and it follows from the definition that $[T]$ is a polynomial. □

**Corollary 4.2.** *The* $q$*-polynomial* $[T]$ *of* $T$ *does not depend on the planarity structure of* $T$. *It only depends on the reduced rooted tree underlying* $T$.

For any $m \in \mathbb{N}_{\geqslant 2}$ let

$$\alpha(m) = \frac{1}{m(m-1)} \cdot \binom{q-2}{m-2}.$$

It is a polynomial $\in \mathbb{Q}[q]$ of degree $m-2$.

For any numeric partition $\lambda$ of $n$ with $m$ parts $\lambda_1, \ldots, \lambda_m$, let

$$\beta(\lambda) := \frac{[n-2]!}{[\lambda_1-1]! \cdot \cdots \cdot [\lambda_m-1]!}.$$

It is a polynomial of degree $\binom{n-2}{2} - \sum_{i=1}^{m} \binom{\lambda_i-1}{2}$ in $\mathbb{Z}[q]$ which is primitive [5, Section 6].

Let $T \in \mathbf{PRT}(n)$, $n \geqslant 2$. Define for $m \geqslant 2$:

$$\nu_T(m) := \sharp\{\eta \in T^0:\ \mathrm{ar}_T(\eta) = m\}$$

is the number of nodes of $T$ of arity $m$.

Let $\mathrm{Par}_m$ denote the set of numeric partitions with $m$ parts. For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathrm{Par}_m$, let

$$\mu_T(\lambda) := \sharp\{\eta \in T^0:\ \mathrm{ar}_T(\eta) = m \text{ and } \mathrm{par}_T(\eta) = \lambda\}$$

where $\mathrm{par}_T(\eta)$ is the unique partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ which is a permutation of $\sharp L(T(\eta_1)), \ldots, \sharp L(T(\eta_m))$ if $\eta_1, \ldots, \eta_m$ are the nodes in $T(\eta)$ which are adjacent to $\eta$. It means there is a permutation $\tau$ of $\{1, 2, \ldots, m\}$ such that $\lambda_{\tau(i)} = \sharp L(T(\eta_i))$ for all $i$.

**Proposition 4.3.** *Let $T \in \mathbf{PRT}(n)$. Then*

$$[T] = \prod_{m=2}^{n} \left( \alpha(m)^{\nu_T(m)} \cdot \prod_{\lambda \in \mathrm{Par}_m} \beta(\lambda)^{\mu_T(\lambda)} \right)$$

*where $\mathrm{Par}_m$ is the set of numeric partitions $\lambda$ with $m$ parts.*

**Proof.** Let $T = T_1 \cdot T_2 \cdot \cdots \cdot T_r$. Then

$$\nu_T(m) = \begin{cases} \sum_{i=1}^{r} \nu_{T_i}(m) & \text{if } m \neq r, \\ 1 + \sum_{i=1}^{r} \nu_{T_i}(r) & \text{if } m = r \end{cases}$$

because $\mathrm{ar}(\rho_T) = r$ where $\rho_T$ is the root of $T$.

Let $\tau = \mathrm{par}_T(\rho_T)$ be the partion of the $r$-tupel $\deg(T_1), \ldots, \deg(T_r)$. Then

$$\nu_T(\lambda) = \begin{cases} \sum_{i=1}^{r} \nu_{T_i}(\lambda) & \text{if } \lambda \neq \tau, \\ 1 + \sum_{i=1}^{r} \nu_{T_i}(\tau) & \text{if } \lambda = \tau. \end{cases}$$

Now for $n = \deg(T)$, $n_i = \deg(T_i)$, we get, by using induction on $n$:

$$\begin{aligned} &\prod_{m=2}^{n} \left( \alpha(m)^{\nu_T(m)} \prod_{\lambda \in \mathrm{Par}_m} \beta(\lambda)^{\mu_T(\lambda)} \right) \\ &\quad = \alpha(r) \cdot \beta(\tau) \prod_{i=1}^{n} \prod_{m=2}^{n} \left( \alpha(m)^{\nu_T(m)} \prod_{\lambda \in \mathrm{Par}_m} \beta(\lambda)^{\mu_T(\lambda)} \right) \\ &\quad = \left( \prod_{i=1}^{r} [T_i] \right) \cdot \frac{1}{r(r-1)} \binom{q-2}{r-2} \cdot \frac{[n-2]!}{[n_1 - 1]! \cdot \cdots \cdot [n_r - 1]!} = [T] \end{aligned}$$

by the recursive definition of $[T]$ above. $\square$

**Example 4.4.** (i) $[x^m] = [m-2]!\frac{1}{m(m-1)}\binom{q-2}{m-2}$ for $m \geqslant 2$. The planar tree $x^m$ is sometimes called the $m$-corolla or $m$-corona, see [9].

(ii) If $T = T_1 \cdot T_2$, $n_i = \deg(T_i)$, $n = \deg(T) = n_1 + n_2$, then

$$[T] = \begin{bmatrix} n-2 \\ n_1 - 1 \end{bmatrix} \cdot \frac{1}{2}[T_1][T_2].$$

(iii) If $T = T_1 \cdot T_2 \cdot T_3$, $n_i = \deg(T_i)$, $n = n_1 + n_2 + n_3$, then

$$[T] = \frac{[n-2]!}{[n_1-1]![n_2-1]![n_3-1]!} \cdot \frac{1}{2 \cdot 3}(q-2)[T_1][T_2][T_3].$$

(iv) If $\deg(T) = n - 1$, then

$$[x \cdot T] = \frac{1}{2}[T].$$

Let $T = (x)_n := x \cdot (x)_{n-1}$ for $n > 1$ and $(x)_1 = x$. It is sometimes called right comb of degree $n$. Then

$$\big[(x)_n\big] = (1/2)^n.$$

## 5. Generic exponential

Let

$$\exp(q, x) := 1 + \sum_{t \in \mathbf{PRT}} \frac{[t]}{[\deg(t) - 1]!} t(x).$$

It is a power series in $\mathbb{Q}(q)\{\{x\}\}_\infty$ where $\mathbb{Q}(q)$ is the field of rational functions over $\mathbb{Q}$ in a single variable $q$. It will be refered to as generic exponential series. Denote by $a(q, t)$ the coefficient of $\exp(q, x)$ with respect to $t \in \mathbf{PRT}$. Then

$$a(q, t) = \frac{[t]}{[\deg(t) - 1]!}.$$

One obtains the recursive formula

$$\big(q^n - q\big)a(q, t) = \binom{q}{m} a(q, t_1) \cdot \cdots \cdot a(q, t_m)$$

if $t = t_1 \cdot \cdots \cdot t_m$ has arity $m$ and $n = \deg(t)$.

**Proposition 5.1.** *Let $k \in \mathbb{N}$, $k \geqslant 2$, and let $\exp_k(x)$ be the $k$-ary exponential series. Then*

$$\exp(k, x) = \exp_k(x).$$

**Proof.** This follows immediately from the definition of $[t]$ and $[n-1]!$ and the recursive definition of $\mathrm{coeff}_t(\exp_k(x))$ in the proof of Proposition 3.1.

More precisely: let $\hat{a}_k(t) = [n-1]!(k) \cdot a_k(t)$ where $a_k = \mathrm{coeff}_t(\exp_k(x))$. From the proof of Proposition 3.1 we have

$$a_k(t) = \frac{m(m-1)\binom{k-2}{m-2}}{[n-1]!(k)} \cdot a_k(t_1) \cdot \cdots \cdot a_k(t_m)$$

if $t = t_1 \cdot t_2 \cdot \cdots \cdot t_m$, because

$$[n-1]!(k) = \frac{k^{n-1}-1}{k-1}.$$

Thus

$$\begin{aligned}\hat{a}_k(t) &= [n-1]!(k) \cdot \frac{1}{m(m-1)}\binom{k-2}{m-2}\frac{\hat{a}_k(t_1)}{[n_1-1]!(k)} \cdot \cdots \cdot \frac{\hat{a}_k(t_m)}{[n_m-1]!(k)}\\ &= \frac{[n-1]!}{[n_1-1]! \cdots [n_m-1]!}(k)\hat{a}_k(t_1) \cdot \cdots \cdot \hat{a}_k(t_m).\end{aligned}$$

By induction on $\deg(t)$ we get that

$$\hat{a}_k(t) = \frac{[t]}{[n-1]!}(k). \qquad \square$$

**Proposition 5.2.** *For $k, m, n \in \mathbb{N}_{\geqslant 2}$ let $f_k(m,n) = k + \frac{\binom{k}{m}}{\binom{q}{m}}(q^n - q) \in \mathbb{Q}(q)$. Then*

(i) *$f_k(m,n)(k) = k^n$, where $f_k(m,n)(k)$ is the value of the rational function $f_k(m,n)$ at the place $k$.*
(ii) $\mathrm{coeff}_t(\exp(q,x)^k) = f_k(m,n) \cdot \mathrm{coeff}_t(\exp(q,x))$, *if $t$ is of arity $m$ and degree $n$.*

**Proof.** (1) Apparently $f_k(m,n)(k) = k + (k^n - k) = k^n$.

(2) Let $E = \exp(q,x) = 1 + F$. Then

$$E^k = 1 + k \cdot F + \sum_{r=2}^{k} \binom{k}{r} F^r.$$

If $t = t_1 \cdot \cdots \cdot t_m$, $\deg(t) = n$, then

$$\mathrm{coeff}_t\bigl(E^k\bigr) = k \cdot \mathrm{coeff}_t(F) + \binom{k}{m}\mathrm{coeff}_{t_1}(E) \cdot \cdots \cdot \mathrm{coeff}_{t_m}(E).$$

Now $\mathrm{coeff}_t(E) = a(q,t)$ and $\binom{q}{m} a(q,t_1) \cdot \cdots \cdot a(q,t_m) = \binom{n}{q-q} a(q,t)$. Thus

$$\mathrm{coeff}_t\left(E^k\right) = \left(k + \frac{\binom{k}{m}}{\binom{q}{m}}\left(q^n - q\right)\right) a(q,t). \qquad \square$$

**Remark 5.3.** Let $E_{m,n} = \sum_{t \in \mathbf{PRT}_m(n)} a(q,t) \cdot t(x)$ and

$$\left(E^k\right)_{m,n} = \sum_{t \in \mathbf{PRT}_m(n)} \mathrm{coeff}_t\left(E^k\right) t(x).$$

Then it follows from Proposition 5.2 that

$$\left(E^k\right)_{m,n} = f_k(m,n) \cdot E_{m,n}.$$

One can consider these formulas as a functional equation generalizing the classical formula $(\exp_k(x))^k = \exp_k(kx)$.

**Proposition 5.4.**

$$\frac{\partial}{\partial x}\left(\exp(q,x)\right) = \exp(q,x).$$

**Proof.** Let

$$f(q,x) = \exp(q,x) - \frac{\partial}{\partial x}\left(\exp(q,x)\right).$$

Then $f(k,x) = 0$ for any $k \in \mathbb{N}_{\geqslant 2}$. Also

$$f(q,x) = \sum_{t \in \mathbf{PRT}} b(q,t) \cdot t(x)$$

and $b(q,t)$ is a rational function for $t \in \mathbf{PRT}$.

As $0 = f(k,x) = \sum_{t \in \mathbf{PRT}} b(k,t)t(x)$ one gets that $b(k,t) = 0$ for all $t \in \mathbf{PRT}$ and all $k \in \mathbb{N}_{\geqslant 2}$. As $b(q,t)$ is a rational function with an infinite number of zeros, it must be equal to 0. $\square$

**Proposition 5.5.**

$$\lim_{q \to \infty} \exp(q,x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}.$$

**Proof.** (1) For any $f \in \mathbb{Q}(q)$, $f \neq 0$, we denote by

$$(q - \mathrm{ord}_\infty)(f)$$

the order of $f$ at the place $\infty$. If $f = f_1/f_2$ and $f_i$ are polynomials in $\mathbb{Q}[q]$, then $(q - \mathrm{ord}_\infty)(f) = \deg(f_2) - \deg(f_1)$. Thus

$$(q - \mathrm{ord}_\infty)\left(\frac{\binom{q}{m}}{q^n - q}\right) = n - m$$

for $m, n \in \mathbb{N}_{\geqslant 2}$. It follows by induction on $n = \deg(t)$ that

$$(q - \mathrm{ord}_\infty)\big(a(q,t)\big) \geqslant 0$$

for all $t \in \mathbf{PRT}$, because if $t = t_1 \cdot t_2 \cdot \cdots \cdot t_m$, then

$$(q - \mathrm{ord}_\infty)\big(a(q,t)\big) = (q - \mathrm{ord}_\infty)\left(\frac{\binom{q}{m}}{q^n - q}\right) \cdot (q - \mathrm{ord}_\infty)\left(\prod_{i=1}^m a(q,t_i)\right)$$

and

$$(q - \mathrm{ord}_\infty)\big(a(q,t)\big) > 0$$

if $n > m$.

But $n \geqslant m$ always and $n = m$ if and only if $t = x^m$ is the $m$-corolla. Thus $\lim_{q\to\infty} a(q,t) = 0$ for all $t \in \mathbf{PRT} - \{x^m\colon m \geqslant 2\}$. Now $a(q,x^m) = \binom{q}{m}/(q^m - q)$ and

$$\begin{aligned} a\big(q,x^m\big)(\infty) &= \left(\text{leading coefficient of } \binom{q}{m}\right) \Big/ \big(\text{leading coefficient of } q^m - q\big) \\ &= \frac{1}{m!}. \qquad \square \end{aligned}$$

**Proposition 5.6.**

$$\exp(0,x) = 1 + \sum_{t \in \mathbf{PRT}} \frac{(-1)^{\bar{n}(t)}}{\prod_{\eta \in t^0}(\mathrm{ar}_t(\eta))} t(x)$$

*where $\bar{n}(t)$ is the number of edges of $t$.*

**Proof.** (1) For $j \in \mathbb{N}_{\geqslant 1}$, the value $[j](0)$ of $[j]$ at the place 0 is equal to 1 and thus also $[j]!(0) = 1$ and $\beta(\lambda)(0) = 1$. For $m \in \mathbb{N}_{\geqslant 2}$, we get

$$\alpha(m)(0) = \frac{1}{m(m-1)} \cdot \binom{-2}{m-2} = (-1)^m \frac{1}{m}.$$

From Proposition 4.3 it follows that $[t](0) = \prod_{m=2}^{m} ((-1)^m \frac{1}{m})^{\nu_t}(m)$ which is obviously equal to

$$\prod_{\eta \in t^{(\mathrm{in})}} (-1)^{\mathrm{ar}(\eta)} \frac{1}{\mathrm{ar}_t(\eta)} = (-1)^{\bar{n}(t)} \cdot \frac{1}{\prod_{\eta \in t^{(\mathrm{in})}}} \mathrm{ar}_t(\eta).$$

(2) As $\mathrm{coeff}_t(\exp(0,x)) = a(0,t) = \frac{[t](0)}{[n-1]!(0)} = [t](0)$ the proof of the proposition follows. □

**Proposition 5.7.**

$$\exp(1,x) = 1 + \sum_{t \in \mathbf{PRT}} \prod_{\eta \in t^{(\mathrm{in})}} \frac{(-1)^{\mathrm{ar}_t(\eta)}}{\mathrm{ar}_t(\eta)(\mathrm{ar}_t(\eta)-1) \cdot (\deg(t(\eta))-1)} \cdot t(x)$$

*where $t^{(\mathrm{in})}$ denotes the set of inner nodes $\eta$ of $t$.*

**Proof.** If $t = t_1 \cdot \cdots \cdot t_m$, then

$$a(q,t) = \frac{1}{m(m-1)} \cdot \frac{\binom{q-2}{m-2}}{[n-1]!} a(q,t_1) \cdot a(q,t_m)$$

and thus

$$a(1,t) = \frac{1}{m(m-1)} \frac{(-1)^{m-2}}{n-1} a(1,t_1) \cdots a(1,t_m).$$

It follows that

$$a(1,t) = \prod_{\eta \in t^0} \frac{(-1)^{\mathrm{ar}_t(\eta)}}{\mathrm{ar}_t(\eta)(\mathrm{ar}_t(\eta)-1) \cdot (\deg(t(\eta))-1)}. \qquad \square$$

## 6. On properties of coefficients of exponentials

Let $e^x$ be the classical Euler exponential series which is the Taylor expansion of the exponential function at 0.

**Proposition 6.1.** *The classical series associated with* $\exp(q,x)$ *is* $e^x$.

**Proof.** (1) We claim that for any $f \in \mathbb{Q}(q)\{\{x\}\}_\infty$ we have

$$\mathrm{can}\left(\frac{\partial}{\partial x}(f)\right) = \frac{\partial}{\partial x}\big(\mathrm{can}(f)\big).$$

First this is proved for monomials by induction on the degree. By standard methods it follows for polynomials and also for power series, as $\frac{\partial}{\partial x}$ is a continuous operator with respect to the topology induced by ord.

(2) From (1) it follows that $f = \mathrm{can}(\exp(q,x))$ is a series with $\frac{\partial}{\partial x}(f) = f$ and $\mathrm{ord}(f - (1+x)) \geqslant 2$. It is known from elementary calculus that one can conclude that $f = e^x$. $\square$

The exponential series gives rise to a sequence of summation formulas for which a combinatorial interpretation would be of interest.

**Proposition 6.2.**

$$\sum_{t \in \mathbf{PRT}(n)} [t] = \frac{[n-1]!}{n!}.$$

**Proof.** $\mathrm{can}(t) = x^n$ for any $t \in \mathbf{PRT}(n)$ and $\mathrm{can}(\exp(q,x)) = e^x$. Thus

$$\sum_{t \in \mathbf{PRT}(n)} a(q,t) = \frac{1}{n!}$$

for $n \geqslant 1$. The proposition follows as $a(q,t) = \frac{[t]}{[n-1]!}$. $\square$

Let $v \in \mathbf{RT}$ and $\mathbf{PRT}(v)$ be the set of all $t \in \mathbf{PRT}$ whose underlying tree is equal to $v$. It follows from Corollary 4.2 that $a(q,t)$ are all equal for $t \in \mathbf{PRT}(v)$. This common value is also denoted by $a(q,v)$. Let $\{v\}(x) = \sum_{t \in \mathbf{PRT}(v)} t(x)$. Then obviously

$$\exp(q,x) = 1 + \sum_{v \in \mathbf{RT}} a(q,v) \cdot \{v\}(x).$$

**Corollary 6.3.** *For $v \in \mathbf{RT}$ denote by $\pi(v)$ the number of $t \in \mathbf{PRT}$ whose underlying tree is $v$.*

(i) $\sum_{v \in \mathbf{RT}(n)} \pi(v) \cdot [v] = \frac{[n-1]!}{n!}$.
(ii) $\sum_{v \in \mathbf{RT}(n)} \pi(v) \cdot [v](0) = \frac{1}{n!}$.
(iii) $\sum_{v \in \mathbf{RT}(n)} \pi(v) \cdot [v](1) = \frac{1}{n}$.

**Proof.** It is immediate by Propositions 6.1, 5.6 and 5.7, as $[n-1]!(1) = (n-1)!$. $\square$

**Proposition 6.4.** *Let $t \in \mathbf{PRT}$ and assume that there is a node in $t$ of arity $m$. Then* $\mathrm{coeff}_t(\exp_k(x)) = 0$ *for all $k < m$.*

**Proof.** Let $\alpha(m) = \frac{1}{m(m-1)}\binom{q-2}{m-2}$ as in Proposition 4.3. If $2 \leqslant k < m$, then $\binom{k-2}{m-2} = 0$. It follows from Proposition 4.3 that $[t](k) = 0$. $\square$

Table 1
$q$-polynomials for trees of degree $\leqslant 5$. $R$ denotes a tree in **RT**, $\pi(R)$ is the number of planar trees in **PRT** whose underlying tree is $R$, $[R]$ is the $q$-polynomial of $R$ and $[R](0)$ the value of $R$ at the place 0

| $\deg(R)$ | $R$ | $\pi(R)$ | $[R]$ | $[R](0)$ |
|---|---|---|---|---|
| 2 | $x^2$ | 1 | $\frac{1}{2}$ | |
| 3 | $x^3$ | 1 | $\frac{1}{3!}(q-2)$ | $-\frac{1}{3}$ |
| | $x^2 \cdot x$ | 2 | $\frac{1}{4}$ | |
| 4 | $x^4$ | 1 | $\frac{1}{4!}(q+1)(q-2)(q-3)$ | $\frac{1}{4}$ |
| | $x^2 \cdot x \cdot x$ | 3 | $\frac{1}{3\cdot 4}(q+1)(q-2)$ | $-\frac{1}{3!}$ |
| | $x^3 \cdot x$ | 2 | $\frac{1}{3\cdot 4!}(q-2)$ | $-\frac{1}{36}$ |
| | $(x^2 \cdot x) \cdot x$ | 4 | $\frac{1}{8}$ | |
| | $x^2 \cdot x^2$ | 1 | $\frac{1}{8}(q+1)$ | $\frac{1}{8}$ |
| 5 | $x^5$ | 1 | $\frac{1}{5!}(q+1)(q^2+q+1)(q-2)(q-3)(q-4)$ | $-\frac{1}{5}$ |
| | $x^2 \cdot x \cdot x \cdot x$ | 4 | $\frac{1}{2\cdot 4!}(q+1)(q^2+q+1)(q-2)(q-3)$ | $-\frac{1}{8}$ |
| | $x^3 \cdot x \cdot x$ | 3 | $\frac{1}{3!5!}(q^2+q+1)(q-2)^2$ | $-\frac{1}{3!\cdot 5}$ |
| | $(x^2 \cdot x) \cdot x \cdot x$ | 6 | $\frac{1}{4!}(q^2+q+1)(q-2)$ | $-\frac{2}{4!}$ |
| | $x^2 \cdot x^2 \cdot x$ | 3 | $\frac{1}{4!}(q^2+q+1)(q-2)$ | $\frac{2}{4!}$ |
| | $x^3 \cdot x^2$ | 2 | $\frac{1}{4!}(q^2+q+1)(q-2)$ | $-\frac{2}{4!}$ |
| | $(x^2 \cdot x) \cdot x^2$ | 4 | $\frac{1}{2\cdot 4!}(q^2+q+1)(q-2)$ | $-\frac{1}{4!}$ |
| | $x^4 \cdot x$ | 2 | $\frac{1}{2}\frac{1}{4!}(q+1)(q-2)(q-3)$ | $\frac{1}{8}$ |
| | $(x^2 \cdot x \cdot x) \cdot x$ | 6 | $\frac{1}{4!}(q+1)(q-2)$ | $-\frac{2}{4!}$ |
| | $(x^3 \cdot x) \cdot x$ | 4 | $\frac{1}{4!}(q-2)$ | $-\frac{2}{4!}$ |
| | $((x^2 \cdot x) \cdot x) \cdot x$ | 8 | $\frac{1}{16}$ | |
| | $(x^2 \cdot x^2) \cdot x$ | 2 | $\frac{1}{16}(q+1)$ | $\frac{1}{16}$ |

## 7. Further work

It seems to me that the results of this article are only the beginning of an area of research about non-associative calculus. However it is difficult to predict at this stage the scope of the more relevant possible discoveries. It appears to me that progress in the following problem areas could lead to interesting developments.

*Area 1: Baker–Campbell–Hausdorff formula*

Let $H_r(q_0,\ldots,q_r;x_1,\ldots,x_r) := \log(q_0, \exp(q_1,x_1)\cdot \cdots \cdot \exp(q_r,x_r))$. It is a general non-associative Hausdorff series. The case $r=2$, $q_i=2$ for all $i$ was studied in [4, Section 6]. Is there an analogue to the formula of Baker–Campbell–Hausdorff in the general case and how does it look like?

*Area 2: Sine, cosine and other functions*

Let $K = \mathbb{Q}(i)$, $i = \sqrt{-1}$. Then

$$\cos(q, x) = 1/2\big(\exp(q, ix) + \exp(q, -ix)\big),$$
$$\sin(q, x) = 1/(2i)\big(\exp(q, ix) - \exp(q, -ix)\big)$$

are power series in $K(q)\{\{x\}\}_\infty$. What are the fundamental properties of these and other trigonometric series such as arc sin which is the compositional inverse of sin? Find other classical functions for which there are non-associative deformations.

*Area 3: Arithmetical properties of coefficients of* exp *and moduli aspects*

Can one find combinatorial proofs for some of the summation formulas in Corollary 6.3 by looking at **PRT** as the moduli space of planar rooted trees? What are the arithmetic properties of the polynomials $[t]$ and the integers $[t](k)$ for $k \in \mathbb{N}$. For $[t](2)$ a result was obtained in [3] in which the Mersenne factorial quotient was involved.

*Area 4: Exponential series* $\exp_v(x)$ *for trees* $v \in \mathbf{PRT}$

There is a further generalization of exponential series as follows: Let $v \in \mathbf{PRT}(k)$ and $f \in \mathbb{Q}\{\{x\}\}_\infty$. One can substitute $f$ for $x$ into $v$ to obtain $v(f)$. More formally if $v = x$, then $x(f) = f$ and if $v = v_1 \cdot v_2 \cdot \cdots \cdot v_m$, then $v(f) = v_1(f) \cdot \cdots \cdot v_m(f)$.

One can show that there is a unique power series $E_v(x) \in \mathbb{Q}\{\{x\}\}_\infty$ such that

$$v\big(E_v(x)\big) = E_v(kx),$$
$$\operatorname{ord}\big(E_v(x) - (1 + x)\big) \geqslant 2.$$

If $v = x^k$, then $E_{x^k}(x) = \exp_k(x)$.

Obviously $(\exp_2(x) \cdot \exp_2(x))(\exp_2(x) \cdot \exp_2(x)) = (x^2 \cdot x^2)(\exp(x)) = \exp_2(4x)$ which shows $E_{x^2 \cdot x^2}(x) = \exp_2(x)$.

One can prove that

$$v(1 + x) = \sum_{i=0}^{n} \frac{v^{(i)}(x)}{i!}$$

where $v^{(i)}(x) = (\frac{\mathrm{d}}{\mathrm{d}x})^i(v)$ is the $i$th derivative of $v$. If $E_v(x) = 1 + F$, then $v(E_v(x)) = 1 + \sum_{i=0}^{n-1} \frac{v^{(i)}(F)}{i!}$ and the coefficients of $E_v(x)$ can be computed as in the proof of Proposition 3.1.

What are general relations among the variety of series $\{E_v(x)\colon v \in \mathbf{PRT}\}$?

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